NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 1)

2011 HSC Course Assessment Task 4

General instructions

- Working time – 50 min.
- **Commence each new question on a new page**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M3A – Mr Lam
- ◯ 12M3B – Mr Weiss
- ◯ 12M3C – Mr Lin
- ◯ 12M4A – Mr Fletcher/Mrs Collins
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mrs Collins/Mr Rezcallah /Mr Lin

STUDENT NUMBER ....   **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	Total	%
MARKS	/10	/11	/12	/33	

Question 1 (10 Marks) Commence a NEW page. **Marks**

(a) The acceleration of a particle moving on the x axis is given by $\ddot{x} = 4x + 2$ where x is its displacement from the origin after t seconds. Initially, the particle was at the origin and had a velocity of $1\,\mathrm{ms}^{-1}$.

i. Show that its velocity at any position x is $v = 2x + 1$. **2**

ii. Find the time taken by the particle to reach a velocity of $9\,\mathrm{ms}^{-1}$. **3**

(b) Find the term independent of x in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{12}$. **2**

(c) By referring to the expansion of $(1+x)^n$, show that **3**

$$2\binom{n}{2} + 6\binom{n}{3} + 12\binom{n}{4} + \cdots + n(n-1)\binom{n}{n} = n(n-1)2^{n-2}$$

by differentiating and substituting an appropriate value for x.

Question 2 (11 Marks) Commence a NEW page. **Marks**

(a) Given **3**

$$(1+x)^{10}(1-x)^{10} = \left(1-x^2\right)^{10}$$

and

$$\binom{n}{r} = \binom{n}{n-r}$$

find the value of

$$\left[\binom{10}{0}\right]^2 - \left[\binom{10}{1}\right]^2 + \left[\binom{10}{2}\right]^2 - \left[\binom{10}{3}\right]^2 + \cdots + \left[\binom{10}{10}\right]^2$$

(b) Mr Lim borrows \$420 000 to purchase an apartment. The interest rate is 7.2% p.a. reducible, and the loan is to be repaid in equal monthly repayments over 30 years, with interest calculated monthly. Let A_n be the amount owing after the n-th repayment.

i. By writing expressions for A_1 and A_2 or otherwise, show that the amount of each monthly repayment is \$2 850.91. **4**

ii. Due to a new wave of financial turmoils, the bank drops the interest rate to 6% p.a. reducible after 12 months of the loan commencing. Find the new monthly repayment, correct to the nearest cent. The period of the original loan remains at 30 years. **4**

Question 3 (12 Marks) Commence a NEW page. **Marks**

(a) A particle is moving in simple harmonic motion about a fixed point O on a straight line. At time t seconds, its displacement x metres is given by

$$x = \cos 2t - \sin 2t$$

i. Express x in terms of $R\cos(2t + \alpha)$ for some $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

ii. Find the amplitude and period of the motion. **2**

iii. Find the initial position of the particle. **1**

iv. Hence or otherwise, find the time when the particle first returns to its initial position. **2**

(b) A particle if projected from a fixed point O on a horizontal plane at an angle of elevation α with speed V metres per second.

After time t, the horizontal and vertical components of its velocity are

$$\dot{x} = V\cos\alpha \qquad \dot{y} = V\sin\alpha - gt$$

i. Show that the position $P(x, y)$ of the particle at any time as it moves along its path is given by **2**

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\left(1 + \tan^2\alpha\right)$$

ii. If the particle is projected from the origin at an angle of 30°, find the speed required for it to just clear a vertical wall 4 m high and 12 m away from the origin. Give your answer correct to 2 decimal places, and take $g = 10\,\text{ms}^{-2}$. **3**

End of paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right) + C$$

NOTE: $\ln x = \log_e x, \ x > 0$

Suggested Solutions

Question 1

(a) i. (2 marks)

✓ [1] for finding $v^2 = 4x^2 + 4x + 1$.

✓ [1] for justifying why $v = 2x + 1$.

$$\begin{aligned} \ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x + 2 \\ \therefore \frac{1}{2}v^2 &= \int 4x + 2\, dx \\ &= 2x^2 + 2x + C \\ v^2 &= 4x^2 + 4x + D \end{aligned}$$

When $t = 0$, $x = 0$ and $v = +1$

$$\begin{aligned} \therefore 1 &= D \\ \therefore v^2 &= 4x^2 + 4x + 1 \end{aligned}$$

As v is initially positive,

$$\therefore v = \sqrt{(2x+1)^2} = 2x + 1$$

ii. (3 marks)

✓ [1] for $t = \frac{1}{2}\ln(2x+1)$.

✓ [2] for substituting $v = 9$ and obtaining $t = \ln 3$.

$$\begin{aligned} v = 9 &= 2x + 1 \\ \therefore 2x &= 8 \\ x &= 4 \\ \frac{dx}{dt} &= 2x + 1 \\ \frac{dx}{2x+1} &= 1\, dt \end{aligned}$$

Integrating both sides,

$$\frac{1}{2}\log_e(2x+1) = t + C$$

When $t = 0$, $x = 0$,

$$\begin{aligned} \frac{1}{2}\log_e 1 &= 0 + C \\ \therefore C &= 0 \\ \therefore t &= \frac{1}{2}\log_e(2x+1) \end{aligned}$$

When $v = 9$, $x = 4$ (from the start of this part)

$$\begin{aligned} \therefore 4 &= \frac{1}{2}\left(e^{2t} - 1\right) \\ 8 &= e^{2t} - 1 \\ e^{2t} &= 9 \\ e^t &= 3 \\ \therefore t &= \log_e 3 \end{aligned}$$

(b) (2 marks)

✓ [1] for obtaining the typical term in the expansion, $T_k = \binom{12}{k}\left(x^2\right)^{12-k}\left(x^{-1}\right)^k$.

✓ [1] for $T_8 = \binom{12}{8}$ $(= 495)$.

$$\left(x^2 + \frac{1}{x}\right)^{12}$$

A typical term in this expansion is

$$\begin{aligned} T_k &= \binom{12}{k}\left(x^2\right)^{12-k}\left(x^{-1}\right)^k \\ &= \binom{12}{k}x^{24-2k}x^{-k} \\ &= \binom{12}{k}x^{24-3k} \end{aligned}$$

The term independent of x occurs when

$$\begin{aligned} 24 - 3k &= 0 \\ \therefore 3k &= 24 \\ k &= 8 \\ \therefore T_8 &= \binom{12}{8} = 495 \end{aligned}$$

(c) (3 marks)

✓ [2] for differentiating correctly (twice)

✓ [1] for substituting $x = 1$ correctly.

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$$

Differentiating,

$$
\begin{aligned}
&n(1+x)^{n-1}\\
&= \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2\\
&\quad + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}
\end{aligned}
$$

Differentiating again,

$$
\begin{aligned}
&n(n-1)(1+x)^{n-2}\\
&= 2\binom{n}{2}x + 3\times 2\binom{n}{3}x + 4\times 3\binom{n}{4}x^2\\
&\quad + \cdots + n(n-1)\binom{n}{n}x^{n-2}
\end{aligned}
$$

Substituting $x=1$,

$$
\begin{aligned}
&n(n-1)2^{n-2}\\
&= 2\binom{n}{2} + 6\binom{n}{3} + 12\binom{n}{4}\\
&\quad + \cdots + n(n-1)\binom{n}{n}
\end{aligned}
$$

Question 2

(a) (3 marks)

- ✓ [1] for recognising x^{10} is the required term.
- ✓ [1] for using $\binom{n}{k} = \binom{n}{n-k}$ on the left to simplify expression.
- ✓ [1] for -252. Relevant working must be shown to obtain [3].

$$
\underbrace{(1+x)^{10}}_{T_k}\underbrace{(1-x)^{10}}_{T_r} = \underbrace{\left(1-x^2\right)^{10}}_{T_m}
$$

$$
T_k = \binom{10}{k}x^k \qquad T_r = \binom{10}{r}(-1)^r x^r
$$

$$
\begin{aligned}
T_m &= \binom{10}{m}(-1)^m\left(x^2\right)^m\\
&= \binom{10}{m}(-1)^m x^{2m}\\
x^k x^r &= x^{2m} = x^{10}
\end{aligned}
$$

As the expansion results in a polynomial of degree 20, yet the expansion only contains $\binom{10}{j}$, hence find the term with x^{10}

$$
\begin{aligned}
T_k \times T_r &= \binom{10}{k}\binom{10}{r}x^k(-1)^r x^r\\
&= \binom{10}{k}\binom{10}{r}x^{k+r}(-1)^r
\end{aligned}
$$

On the right,

$$
\begin{aligned}
2m &= 10\\
\therefore m &= 5\\
\therefore T_5 &= \binom{10}{5}(-1)^5 x^{10} = -\binom{10}{5}x^{10}\\
&= -252x^{10}
\end{aligned}
$$

On the left, the term with x^{10} occurs when

$$
\therefore k + r = 10
$$

k	r
10	0
9	1
⋮	⋮
0	10

The term with x^{10} on the left is thus

$$
\begin{aligned}
&\binom{10}{0}\binom{10}{10} + (-1)\binom{10}{1}\binom{10}{9}\\
&\quad + \binom{10}{2}\binom{10}{8} + (-1)\binom{10}{3}\binom{10}{7}\\
&\quad + \cdots + (-1)\binom{10}{9}\binom{10}{1} + \binom{10}{10}\binom{10}{0}
\end{aligned}
$$

By symmetry,

$$
\binom{10}{0} = \binom{10}{10} \qquad \binom{10}{1} = \binom{10}{9}
$$

Hence

$$
\begin{aligned}
&\left[\binom{10}{0}\right]^2 - \left[\binom{10}{1}\right]^2 + \left[\binom{10}{2}\right]^2\\
&\quad - \left[\binom{10}{3}\right]^2 + \cdots + \left[\binom{10}{10}\right]^2 = -252
\end{aligned}
$$

(b) i. (4 marks)

✓ [1] for A_1 and A_2.
✓ [1] for summing GP.
✓ [1] for $A_{360} = 0$.
✓ [1] for $M = 2\,850.91$.

- $P = \$420\,000$,
- $r = \frac{0.072}{12} = 0.006$ p.m.

Let the amount outstanding at the end of the k-th month be A_k.

- $A_1 = P \times 1.006 - M$
- $A_2 = A_1 \times 1.006 - M$
  $= (1.006P - M)1.006 - M$
  $= 1.006^2 P - M(1 + 1.006)$
- $A_n = 1.006^n P - M\left(1 + 1.006 + \cdots + 1.006^{n-1}\right)$

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{1(1.006^n - 1)}{1.006 - 1} \\ &= 1\,269.23 \end{aligned}$$

When the loan is repaid in $30 \times 12 = 360$ months, $A_{360} = 0$.

$$\begin{aligned} 1.006^{360} \times 500 &= 1\,269.23M \\ \therefore M &= \frac{1.006^{360} \times 500}{1\,269.23} = \$2\,850.91 \end{aligned}$$

ii. (4 marks)

✓ [1] for finding $A_{12} = 415\,895.38$.
✓ [1] for finding new A_n
✓ [1] for evaluating sum of GP.
✓ [1] for final answer.

Find the amount owing after 12 months:

$$\begin{aligned} A_{12} &= 420\,000 \times 1.006^{12} \\ &\quad - 2\,850.91\left(\frac{1.006^{12} - 1}{1.006 - 1}\right) \\ &= 415\,895.38 \end{aligned}$$

Now in the 13th month, the interest rate changes to $r = \frac{0.06}{12} = 0.005$ p.m. with monthly repayment K:

$$\begin{aligned} A_{13} &= A_{12} \times 1.005 - K \\ A_{14} &= A_{13} \times 1.005 - K \\ &= (A_{12} \times 1.005 - K) \times 1.005 - K \\ &= 1.005^2 A_{12} - K(1 + 1.005) \\ &\vdots \\ A_{360} &= 1.005^{360-12} A_{12} \\ &\quad - K(1 + 1.005 + \cdots + 1.005^{360-12-1}) \end{aligned}$$

Evaluating the sum of GP with $n = 360 - 12 = 348$

$$\begin{aligned} S_{348} &= \frac{1(1.005^{348} - 1)}{1.005 - 1} \\ &= 934.54 \end{aligned}$$

At the 360th month, the loan is repaid. $A_{360} = 0$

$$\begin{aligned} 0 &= 1.005^{348} A_{12} - 934.54K \\ \therefore K &= \frac{415\,895.38 \times 1.005^{348}}{934.54} \\ &= 2\,524.50 \end{aligned}$$

Question 3

(a) i. (2 marks)

✓ [1] for $R = \sqrt{2}$.
✓ [1] for $\alpha = \frac{\pi}{4}$.

$$\begin{aligned} x &= \cos 2t - \sin 2t \\ &\equiv R\cos(2t + \alpha) \\ &= R\cos 2t \cos\alpha - R\sin 2t \sin\alpha \end{aligned}$$

Equating coefficients of $\cos 2t$ and $\sin 2t$,

$$\begin{cases} R\cos\alpha = 1 & (1) \\ R\sin\alpha = 1 & (2) \end{cases}$$

(2) ÷ (1)

$$\begin{aligned} \tan\alpha &= 1\\ \therefore \alpha &= \frac{\pi}{4}\\ R\cos\frac{\pi}{4} &= 1\\ R\times\frac{1}{\sqrt{2}} &= 1\\ \therefore R &= \sqrt{2}\\ \therefore x &= \sqrt{2}\cos\left(2t+\frac{\pi}{4}\right) \end{aligned}$$

ii. (2 marks)
- ✓ [1] for $a = \sqrt{2}$.
- ✓ [1] for $T = \pi$.

$$\begin{aligned} \text{amplitude} &= \sqrt{2}\\ T &= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi \end{aligned}$$

iii. (1 mark)

$$\begin{aligned} x &= \sqrt{2}\cos\left(2t+\frac{\pi}{4}\right)\Big|_{t=0}\\ &= \sqrt{2}\times\cos\frac{\pi}{4} = 1 \end{aligned}$$

iv. (2 marks)
- ✓ [1] for $2t + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}\cdots$.
- ✓ [1] for $t = \frac{3\pi}{4}$.

$$\begin{aligned} 1 &= \sqrt{2}\cos\left(2t+\frac{\pi}{4}\right)\\ \cos\left(2t+\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\\ 2t+\frac{\pi}{4} &= \frac{\pi}{4}, \frac{7\pi}{4}\cdots\\ 2t &= 0, \frac{3\pi}{2}\cdots\\ t &= 0, \frac{3\pi}{4}\cdots \end{aligned}$$

Hence the particle first returns to $x = 1$ at $t = \frac{3\pi}{4}$.

(b) i. (2 marks)
- ✓ [1] for integrating correctly.
- ✓ [1] for substituting $t = \frac{x}{V\cos\alpha}$ correctly and showing what is required.

$$\begin{cases} \dot{x} = V\cos\alpha\\ \dot{y} = V\sin\alpha - gt \end{cases}$$

Integrating,

$$\begin{cases} x = Vt\cos\alpha + C_1 g\\ y = Vt\sin\alpha - \frac{1}{2}gt^2 + C_2 \end{cases}$$

Initially, $x = 0$ and $y = 0$. Hence $C_1 = C_2 = 0$.

$$\therefore \begin{cases} x = Vt\cos\alpha\\ y = Vt\sin\alpha - \frac{1}{2}gt^2 \end{cases}$$

Remove parameter t to obtain Cartesian equation,

$$\begin{aligned} t &= \frac{x}{V\cos\alpha}\\ y &= V\left(\frac{x}{V\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2\\ &= x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}\\ &= x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha\\ &= x\tan\alpha - \frac{gx^2}{2V^2}\left(1+\tan^2\alpha\right) \end{aligned}$$

ii. (3 marks)
- ✓ [1] for substituting $\alpha = 30°$, $x = 72$ and $y = 4$.
- ✓ [1] for obtaining $v^2 = \frac{960}{\frac{12}{\sqrt{3}}-4}$ or equivalent.
- ✓ [1] for $v = 18.1\,\text{ms}^{-1}$.

$$\alpha = 30° \qquad x = 12 \qquad y = 4$$

$$\begin{aligned} 4 &= 12\tan 30° - \frac{10\times 144}{2V^2}\left(1+\tan^2 30°\right)\\ 4 &= \frac{12}{\sqrt{3}} - \frac{720\left(1+\frac{1}{3}\right)}{V^2}\\ 4 &= \frac{12}{\sqrt{3}} - \frac{960}{V^2}\\ \frac{960}{V^2} &= \frac{12}{\sqrt{3}} - 4\\ V^2 &= \frac{960}{\frac{12}{\sqrt{3}}-4}\\ \therefore V &= \sqrt{\frac{960}{\frac{12}{\sqrt{3}}-4}} = 18.1\,\text{ms}^{-1} \end{aligned}$$